ON THE REFLECTION OF WEAK DISCONTINUITIES FROM THE THROAT OF A LAVAL NOZZLE

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In this paper consideration is given to three-dimensional flows of an ideal gas in Laval nozzles, which have discontinuities in the first derivatives of the velocity components on particular characteristic surfaces. The solutions obtained yield examples of transformations from three-dimensional gas flows into plane-parallel and axial-symmetric flows, and also into other flows in space.

1. Statement of the problem. Let us investigate the flow of an ideal gas in a Laval nozzle which has two planes of symmetry. Let the origin of the cylindrical coordinate system x, r, θ be coincident with the throat of the nozzle and the x-axis be coincident with the axis of the nozzle. We write the equation which determines the flow of the gas in the region of the sonic surface in the form

$$-\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}=0 \qquad (1.1)$$

where ϕ is a potential and

$$\frac{a_*}{\varkappa+1}\frac{\partial\varphi}{\partial x}=v_x, \qquad \frac{a_*}{\varkappa+1}\frac{\partial\varphi}{\partial r}=v_r, \qquad \frac{a_*}{\varkappa+1}\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta}=v_\vartheta$$

where v_x, v_r and v_θ are the velocity perturbations along the x, r, and θ axes. The velocity is equal in magnitude to the critical velocity a_* and is directed along the nozzle axis; κ is the exponent of the Poisson adiabatic.

In Reference (1) the solution of equation (1.1) was obtained which describes the flow in analytical Laval nozzles; this solution will serve as the basis for further investigation:

$$\varphi = \tfrac{1}{2} c_1 x^2 + c_1^2 \left(\tfrac{1}{4} - c_2 \cos 2\vartheta\right) x r^2 + c_1^3 \left(\tfrac{1}{64} - \tfrac{1}{12} c_2 \cos 2\vartheta + c_3 \cos 4\vartheta\right) r^4$$

$$\frac{\varkappa + 1}{a_*} v_x = c_1 x + c_1^2 \left(\tfrac{1}{4} - c_2 \cos 2\vartheta\right) r^2 \tag{1.2}$$

$$\frac{\varkappa + 1}{a_*} v_r = c_1^2 \left(\tfrac{1}{2} - 2c_2 \cos 2\vartheta\right) x r + c_1^3 \left(\tfrac{1}{16} - \tfrac{1}{3} c_2 \cos 2\vartheta + 4c_3 \cos 4\vartheta\right) r^3$$

$$\frac{\varkappa + 1}{a_*} v_\vartheta = 2c_1^2 c_2 x r \sin 2\vartheta + c_1^3 \left(\tfrac{1}{6} c_2 \sin 2\vartheta - 4c_3 \sin 4\vartheta\right) r^3$$

Hereafter we will always assume that $c_1 > 0$. If in the formulas (1.2) $c_1 = c_2 = 0$ is assumed, then we obtain the flow in a round nozzle; assuming $|c_1| = 1/4$, $c_2 = 1/192$ we have the case of plane flow [1-4].

In the supersonic region of the flows considered, i.e. in the region situated downstream of the surface of transition

$$-x = c_1 \left(\tfrac{1}{4} + c_2\right) y^2 + c_1 \left(\tfrac{1}{4} - c_2\right) z^2 \tag{1.3}$$

there corresponds to every point in space its characteristic cone. The curves which envelop these cones form characteristic surfaces. In Reference [1] the solutions of four such characteristics were obtained which possess the same planes of symmetry as the nozzle itself. They pass through its throat and do not have discontinúities:

$$x = \tfrac{1}{16} c_1 [2 - (\delta_2 + \delta_1) + (\delta_2 - \delta_1) \cos 2\vartheta] r^2$$

$$x = \tfrac{1}{16} c_1 [2 \mp (\delta_2 - \delta_1) \pm (\delta_2 + \delta_1) \cos 2\vartheta] r^2 \tag{1.4}$$

$$x = \tfrac{1}{16} c_1 [2 + (\delta_2 + \delta_1) - (\delta_2 - \delta_1) \cos 2\vartheta] r^2$$

where

$$\delta_1 = \sqrt{5 - 16c_2}, \quad \delta_2 = \sqrt{5 + 16c_2}$$

It follows that the solutions of (1.4) exist only for $|c_2| \leqslant 5/16$. Let us denote the characteristic surfaces, which are tangent to the surface of transition (1.3) at the throat of a nozzle and which extend all the way up and down along the stream by c_-^0 and c_+^0, respectively. The first of the formulas (1.4) then gives for c_-^0 an elliptical paraboloid for $|c_2| < 1/4$ c_-^0 for $|c_2| = 1/4$ - c_-^0 a parabolic cylinder and for $|c_2| > 1/4$ a hyperbolic paraboloid. The last of the formulas (1.4) for any admissible c_2 gives for c_+^0 an elliptical paraboloid. Along this surface are propagated the perturbations which originate from the throat of the nozzle at the origin of the coordinates. The second of the formulas (1.4) describes two characteristic surfaces. For $|c_2| \leqslant 1/4$ both form hyperbolic paraboloids; for $|c_2| = 1/4$ one of them will be a c_+^0-parabolic cylinder, and the second a hyperbolic paraboloid; for $|c_2| > 1/4$ one of the surfaces becomes a c_+^0-elliptical paraboloid, while the other remains a hyperbolic paraboloid.

In this manner, characteristic c_-^0-surfaces exist only for $|c_2| < 1/4$ which is a natural consequence of the form of the surface of transition (1.3). For $5/16 < |c_2| < 1/4$ two c_+^0-elliptical paraboloids extend downstream from the throat of the nozzle which are tangent to each other along the curve lying in the planes $y = 0$ and $z = 0$ depending on the sign in the second equation of (1.4).

The outside c_+^0-paraboloid is given by the second formula of (1.4) and the inside paraboloid by the last formula. The parabola formed by the cross-section of the outside c_+^0-paraboloid with the planes $z = 0$ and $y = 0$ correspondingly is the curve which serves as the boundary for the point sources lying in these planes; these sources are within the curve. Perturbations originating from them therefore do not reach the sonic surface. If the sources are placed outside the above parabola, then the perturbations caused by them will reach the surface of transition. For $|c_2| = 5/16$ both c_+^0-elliptical paraboloids coincide. In the plane case, i.e. for $|c_2| = 1/4$ the c_-^0 and also the outside c_+^0-elliptical paraboloids are transformed into c_-^0- and c_+^0-parabolic cylinders. For $|c_2| > 5/16$ no characteristic surface exists which does not touch the sonic surface at any place besides the nozzle center and which extends downstream.

Let us now investigate the propagation of weak discontinuities, i.e. the discontinuities in the first derivatives of the stream velocity components along the characteristic surfaces. In particular we will consider only the singular c^0-characteristic surfaces, i.e. those that touch the surface of transition at the throat of the nozzle. In the general case this problem is reduced to the investigation of all possible continuations of the solution (1.2) of the equation (1.1) for which for $r = 0$ the stream velocity is directed along the nozzle axis into the region downstream of the corresponding characteristic. Of greatest interest here is the problem of the reflection of discontinuities from the throat of the nozzle. This problem is considered in this paper. Let us consider therefore that the undisturbed flow, in which the transition through the velocity of sound takes place, is given by the formulas (1.2). This flow extends upstream from the characteristic surface given by the first of the formulas (1.4). We will denote this region by the number I and assume that within it

$$c_1 = A_1, \quad c_2 = n_1, \quad c_3 = m_1, \quad \Delta_1' = \sqrt{5 - 16n_1}, \quad \Delta_2' = \sqrt{5 + 16n_1}$$

It is evident that the solutions of the equation (1.1) in the region situated downstream of the indicated surface, may be expected to have the form $\phi = r^4 f(\xi, \theta)$, $\xi = x/r^2$. In this way the problem is reduced to the integration of a partial differential equation of the second order with two independent variables:

$$\left(4\xi^2 - \frac{\partial f}{\partial \xi}\right)\frac{\partial^2 f}{\partial \xi^2} - 12\xi \frac{\partial f}{\partial \xi} + \frac{\partial^2 f}{\partial \vartheta^2} + 16f = 0 \tag{1.5}$$

The initial conditions for the equation (1.5) are given along the curve

$$\xi = \tfrac{1}{16} A_1 [2 - (\Delta_2' + \Delta_1') + (\Delta_2' - \Delta_1') \cos 2\vartheta] \tag{1.6}$$

The desired solutions must satisfy the condition

$$v_r = v_\vartheta = 0 \qquad \text{for } r = 0$$

2. Some properties of the equation (1.5). Characteristic curves of the equation (1.5), corresponding to integral (1.2), are given by the equation

$$d\xi^2 + (4\xi^2 - c_1\xi - \tfrac{1}{4} c_1^2 + c_1^2 c_2 \cos 2\vartheta)\, d\vartheta^2 = 0 \tag{2.1}$$

It follows that equation (1.5) is of the hyperbolic type in the region

$$\xi^2 - \tfrac{1}{4} c_1\xi - \tfrac{1}{4} c_1^2 \left(\tfrac{1}{4} - c_2 \cos 2\vartheta\right) < 0$$

and is elliptic if

$$\xi^2 - \tfrac{1}{4} c_1\xi - \tfrac{1}{4} c_1^2 \left(\tfrac{1}{4} - c_2 \cos 2\vartheta\right) > 0$$

Along the curves

$$\xi = \tfrac{1}{8} c_1 \left(1 \pm \sqrt{5 - 16 c_2 \cos 2\vartheta}\right) \tag{2.2}$$

the type of the equation changes. From equation (2.2) follows that for

$$|c_2| > \tfrac{5}{16} \tag{2.3}$$

there exists a region of values θ for which, for any values of ξ, equation (1.5) is an equation of the elliptic type. Inequality (2.3) is the result of the fact that for these values of c_2 no characteristic surface exists touching the surface of transition only at the nozzle throat (see 1).

Let us now note that curve (1.6) is an integral for equation (2.1) and is consequently a characteristic of equation (1.5), i.e. the characteristic surface (1.4) of equation (1.1) has become the characteristic curve of equation (1.5). The problem set out in the preceding paragraph naturally has an infinite number of solutions.

Their general investigation is quite difficult, though equation (1.5) is simpler than the original equation (1.1). In references [2,4] it was determined, however, that for the plane and round nozzles the discontinuities in the first derivatives of the velocity components passing along the c_-^0 characteristics into the throat of the nozzle are generally speaking reflected from the throat along the c_+^0 characteristics in the form of discontinuities in the second and third derivatives.

There exists a unique flow which is finite and in which the discontinuities in the first derivatives are also propagated along the c_+^0 characteristics.

The flows with the said property may be expected to be finite in the general case also. We will establish their structure.

3. Construction of the solution. The discontinuities in the first derivatives of the stream velocity components brought into the nozzle throat along the first of surfaces (1.4) are reflected from it along one of the other three surfaces (1.4). These surfaces may be different for different values of n_1, which follows from the results presented in references [2,4].

The region extending downstream from the particular characteristic along which the "reflected" discontinuities propagate will be marked *II*; and that situated between the two c_-^0 characteristics indicated *III*. The solution in the region *II* and *III* may be expected to appear as before, in form (1.2). Let us assume that in region *II*

$$c_1 = A_2, \qquad c_2 = n_2, \qquad c_3 = m_2, \qquad \Delta_1'' = \sqrt{5 - 16n_2}, \qquad \Delta_2'' = \sqrt{5 + 16n_2}$$

while in region *III*

$$c_1 = A, \qquad c_2 = n, \qquad c_3 = m$$

Along the characteristic surfaces separating the regions *I* and *III* and *II* the values v_x, v_r, and v_θ must coincide. These conditions establish the equations which allow the quantities A, n, m and A_2, n_2, m_2 to be expressed in terms of the values A_1, n_1, m_1 which characterize the undisturbed flow. When writing the desired equations we assume that the weak discontinuities are reflected along the last of surfaces (1.4). Along the c_-^0 characteristic separating regions *I* and *III* we have

$$\frac{1}{4} A_1^2 [2 - (\Delta_2' + \Delta_1')] + A_1^2 = \frac{1}{4} AA_1 [2 - (\Delta_2' + \Delta_1')] + A^2$$

$$\frac{1}{16} A_1^2 (\Delta_2' - \Delta_1') - A_1^2 n_1 = \frac{1}{16} AA_1 (\Delta_2' - \Delta_1') - A^2 n$$

$$\frac{1}{2} A_1^3 [2 - (\Delta_2' + \Delta_1')] - A_1^3 n_1 (\Delta_2' - \Delta_1') + A_1^3 =$$

$$= \frac{1}{2} A^2 A_1 [2 - (\Delta_2' + \Delta_1')] - A^2 A_1 n (\Delta_2' - \Delta_1') + A^3 \qquad (3.1)$$

$$\frac{1}{8}A_1^3 n_1[2-(\Delta_2'+\Delta_1')]-\frac{1}{32}A_1^3(\Delta_2'-\Delta_1')+\frac{1}{3}A_1^3 n_1=$$
$$=\frac{1}{8}A^2A_1n[2-(\Delta_2'+\Delta_1')]-\frac{1}{32}A^2A_1(\Delta_2'-\Delta_1')+\frac{1}{3}A^3n$$
$$\frac{1}{16}A_1^3n_1(\Delta_2'-\Delta_1')-4A_1^3m_1=\frac{1}{16}A^2A_1n(\Delta_2'-\Delta_1')-4A^3m$$
$$\frac{1}{4}A_1^3n_1[2-(\Delta_2'+\Delta_1')]+\frac{1}{3}A_1^3n_1=\frac{1}{4}A_1A^2n[2-(\Delta_2'+\Delta_1')]+\frac{1}{3}A^3n$$

Analogously, along the c_-^0 characteristic separating the regions *III* and *II*, we obtain the second group of equations:

$$\frac{1}{4}A_2^2[2+(\Delta_2''+\Delta_1'')]+A_2^2=\frac{1}{4}AA_2[2+(\Delta_2''+\Delta_1'')]+A^2$$
$$\frac{1}{16}A_2^2(\Delta_2''-\Delta_1'')+A_2^2n_2=\frac{1}{16}AA_2(\Delta_2''-\Delta_1'')+A^2n$$
$$\frac{1}{2}A_2^3[2+(\Delta_2''+\Delta_1'')]+A_2^3n_2(\Delta_2''-\Delta_1'')+A_2^3=$$
$$=\frac{1}{2}A^2A_2[2+(\Delta_2''+\Delta_1'')]+A^2A_2n(\Delta_2''-\Delta_1'')+A^3 \qquad (3.2)$$
$$\frac{1}{8}A_2^3n_2[2+(\Delta_2''+\Delta_1'')]+\frac{1}{32}A_2^3(\Delta_2''-\Delta_1'')+\frac{1}{3}A_2^3n_2=$$
$$=\frac{1}{8}A^2A_2n[2+(\Delta_2''+\Delta_1'')]+\frac{1}{32}A^2A_2(\Delta_2''-\Delta_1'')+\frac{1}{3}A^3n$$
$$\frac{1}{16}A_2^3n_2(\Delta_2''-\Delta_1'')+4A_2^3m_2=\frac{1}{16}A^2A_2n(\Delta_2''-\Delta_1'')+4A^3m$$
$$\frac{1}{4}A_2^3n_2[2+(\Delta_2''+\Delta_1'')]+\frac{1}{3}A_2^3n_2=\frac{1}{4}A^2A_2n[2+(\Delta_2''+\Delta_1'')]+\frac{1}{3}A^3n$$

In system (3.1) there are six equations and only three unknowns. However, it may be shown that quantities A and n determined from the first and the second equation also satisfy the third, the fourth and the sixth equation of this system. The fifth equation serves to determine the value of m. Therefore the over-determination of the system of equations (3.1) is illusory. The same may also be said concerning the equations (3.2).

The first of equations (3.1) is quadratic with respect to the quantity A. Its one root may be determined at once. $A = A_1$. The flow that corresponds to this root pertains to an analytical nozzle and, therefore, we are not interested in it. The second root is given by the equation

$$A=-\frac{1}{4}A_1(6-\Delta_2'-\Delta_1') \qquad (3.3)$$

The value n corresponding to it is determined from the second equation (3.1):

$$n=\frac{48n_1-5(\Delta_2'-\Delta_1')}{2(6-\Delta_2'-\Delta_1')^2} \qquad (3.4)$$

The quantity m is now given by the relationship

$$m = \frac{5(\Delta_2' - \Delta_1')^2 - 16n_1(\Delta_2' - \Delta_1') - 2048m_1}{32(6 - \Delta_2' - \Delta_1')^3} \tag{3.5}$$

Analogously, from equations (3.2) we find

$$A = -\frac{1}{4} A_2 (6 + \Delta_2'' + \Delta_1'') \tag{3.6}$$

$$n = \frac{48n_2 + 5(\Delta_2'' - \Delta_1'')}{2(6 + \Delta_2'' + \Delta_1'')^2} \tag{3.7}$$

$$m = \frac{5(\Delta_2'' - \Delta_1'')^2 + 16n_2(\Delta_2'' - \Delta_1'') - 2048m_2}{32(6 + \Delta_2'' + \Delta_1'')^3} \tag{3.8}$$

Equating quantities (3.3)-(3.5) to the corresponding quantities (3.6)-(3.8), we may obtain values A_2, n_2 and m_2 expressed directly in terms of the parameters of the basic flow.

In exactly the same manner we may investigate the flows when the perturbations, brought into the throat are reflected from it along the c^0 characteristic surfaces given by the second formula of (1.4). As a result we obtain

$$A_1(6 - \Delta_2' - \Delta_1') = A_2(6 \mp \Delta_2'' \pm \Delta_1'') \tag{3.9}$$

$$\frac{48n_1 - 5(\Delta_2' - \Delta_1')}{(6 - \Delta_2' - \Delta_1')^2} = \frac{48n_2 \mp 5(\Delta_2'' + \Delta_1'')}{(6 \mp \Delta_2'' \pm \Delta_1'')^2} \tag{3.10}$$

$$\frac{5(\Delta_2' - \Delta_1')^2 - 16n_1(\Delta_2' - \Delta_1') - 2048m_1}{(6 - \Delta_2' - \Delta_1')^3} = \frac{5(\Delta_2'' + \Delta_1'')^2 \mp 16n_2(\Delta_2'' + \Delta_1'') - 2048m_2}{(6 \mp \Delta_2'' \pm \Delta_1'')^3} \tag{3.11}$$

The signs in formulas (3.9)-(3.11) are to be chosen in accordance with the signs of relationship (1.4).

Relationships (3.1)-(3.11) were obtained by continuing the first derivatives $\partial\phi/\partial x$, $\partial\phi/\partial r$, $\partial\phi/\partial\theta$, which belong to the analytically different solutions along the corresponding characteristics. These conditions express the requirement that the physical quantities must be single-valued. But they also guarantee the continuity of the function ϕ itself on the characteristic surfaces indicated.

The dependence of ratio A_2/A_1 on quantity n_1, given by formulas (3.3), (3.6) and (3.9), is represented in Fig. 1 where $N_1 = 16n_1$.

The dependance of parameter n_2 on the same quantity n_1, described by equations (3.4), (3.7) and (3.10), is given in Fig. 2, where $N = 16n_1$, and $N_2 = 16n_2$. The solid line is for flows for which weak discontinuities are reflected from the nozzle throat along the characteristic surfaces in accordance with the last of formulas (1.4), while the broken line refers to the second of these formulas.

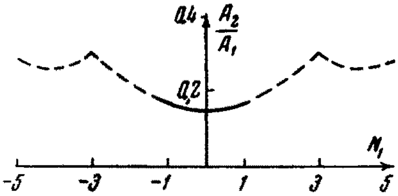

Fig. 1.

It is seen from Fig. 2 that the undisturbed stream which possesses axial symmetry transforms into another axi-symmetric flow. But if we were to distort the basic stream (i.e. change quantity n_1 slightly), then the flow behind the $c_+{}^0$ elliptical paraboloid will increasingly deviate from axial symmetry, while the paraboloid itself will begin to flatten out. For $n_1 \approx \pm\ 0.047$ we have $n_2 = \pm\ 1/4$ i.e. the given three-dimensional flow, which generally speaking approximates to the stream in a round nozzle, transforms into the plane-parallel flow existing inside the $c_+{}^0$ elliptical paraboloid.

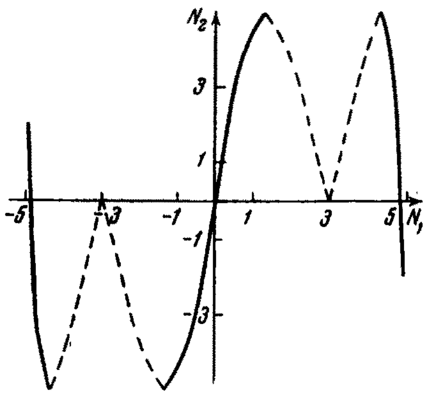

Fig. 2.

For $n_1 \approx \pm\ 0.081$ it is seen that $n_2 = \pm\ 5/16$, and we obtain the flow in which both $c_+{}^0$ paraboloids coincide. If the absolute value of n_1 is increased still further, then the weak discontinuities brought into the throat along the $c_-{}^0$ characteristic are reflected from it along the surface given by the second formula of (1.4). This surface is of the form of the outside $c_+{}^0$ elliptical paraboloid, which proceeds to flatten out with increase of $|n_1|$, and which for $|n_1|$ 0.12 transforms into a $c_+{}^0$ parabolic cylinder with a plane-parallel flow within. For $0.12 < |n_1| < 1/4$ the characteristic surface along which the reflection of weak discontinuities takes place is of the form of a hyperbolic paraboloid, which for

$|n_1|$ 0.19 the resulting flow transforms into an axi-symmetric flow. For $n_1 = \pm$ 1/4 we have $n_2 = \pm$ 1/4, i.e. the plane stream transforms into another plane flow. For $|n_1| > 1/4$ the $c_-{}^0$ characteristic transforms into a hyperbolic paraboloid, while for the $c_+{}^0$ characteristic surface there appears first the outside then for $|n_1| > 0{,}275$ the inside elliptical paraboloid. These two coincide for the latter value of $|n_1|$.

For $|n_1| > 0.12$, one of the characteristic surfaces in which the discontinuities in the second derivatives of the function $\phi(x, r, \theta)$ occur is a hyperbolic paraboloid. In this case the expression "reflection" becomes inexact for these values of n_1 (the exception is the case when $n = \pm$ 1/4).

Note that for $n_1 \approx \pm$ 0.306 we obtain the flow with axial symmetry inside the $c_+{}^0$-paraboloid of revolution. The flows investigated thus furnish examples for the transformation of some of the three-dimensional flows into plane-parallel and axi-symmetrical flows.

From Fig. 1 it follows that the flows in round nozzles possess a minimum coefficient A_2 for the given value A_1, compared with all the flows investigated. By analogy with the results in references [2-4], we may expect the curve of Fig. 1 to define the minimum value of ratio A_2/A_1 which may be obtained for the given values of n_1.

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